Thoroughly Revised & Updated

Engineering Mathematics

For GATE and ESE Examination

- Topic-wise presentation
- Fully Solved with Explanations
- Comprehensive Theory with Examples
- Solved Questions of GATE Examination

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Syllabus

Engineering Mathematics

Linear Algebra: Matrix algebra, systems of linear equations, Eigen values and Eigen vectors.

Calculus: Mean value theorems, theorems of integral calculus, partial derivatives, maxima and minima, multiple integrals, Fourier series, vector identities, line, surface and volume integrals, Stokes, Gauss and Green's theorems.

Differential equations: First order equation (linear and nonlinear), higher order linear differential equations with constant coefficients, method of variation of parameters, Cauchy's and Euler's equations, initial and boundary value problems, solution of partial differential equations: variable separable method.

Analysis of complex variables: Analytic functions, Cauchy's integral theorem and integral formula, Taylor's and Laurent's series, residue theorem, solution of integrals.

Probability and Statistics: Sampling theorems, conditional probability, mean, median, mode and standard deviation, random variables, discrete and continuous distributions: normal, Poisson and binomial distributions.

Numerical Methods: Matrix inversion, solutions of non-linear algebraic equations, iterative methods for solving differential equations, numerical integration, regression and correlation analysis.

CONTENTS

Unit I Linear Algebra

1.1 MATRIX

Matrix is a rectangular array in which elements are arranged in horizontal & vertical lines, where horizontal lines are called rows and vertical lines are called columns.

e.g.,
$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n} = [a_{ij}]_{m\times n}$$

Where, a_{ij} = matrix entry/element of i^{th} row and j^{th} column.

1.2 TRANSPOSE OF MATRIX

Transpose of matrix 'A' is obtained by interchanging rows and columns and it is denoted by A' or A^T.

If $A = [a_{ij}]_{m\times n}$

then, $A^T = A' = [a_{ji}]_{n\times m}$

e.g.,
$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}, \; A' = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

1.3 CONJUGATE OF MATRIX

Conjugate of matrix 'A' is obtained by replace i by –i at every matrix entry, where $i = \sqrt{-1}$ and it is denoted by $\bar{A}$.

If $A = [a_{ij}]_{m\times n}$

then, $\bar{A} = [\bar{a}_{ij}]_{m\times n}$

e.g.,
$$A = \begin{bmatrix} 1+i & -2 & 3+i \\ 4i & 0 & 5 \end{bmatrix}; \; \bar{A} = \begin{bmatrix} 1-i & -2 & 3-i \\ -4i & 0 & 5 \end{bmatrix}$$

1.4 TRANSPOSE CONJUGATE OF MATRIX

Transpose conjugate of matrix is also called as tranjugate of matrix and it is denoted by A^θ.

If $A = [a_{ij}]_{m\times n}$

then, $A^\theta = [\bar{a}_{ji}]_{n\times m} = (\bar{A})^T = (\overline{A^T})$

e.g.,
$$A = \begin{bmatrix} 1+i & -2 & 3+i \\ 4i & 0 & 5 \end{bmatrix}; \; A^\theta = \begin{bmatrix} 1-i & -4i \\ -2 & 0 \\ 3-i & 5 \end{bmatrix}$$

1.5 TYPES OF MATRIX

Let m = total number of rows and n = total number of columns

then order of matrix is $m \times n$.

1.5.1 Rectangular Matrix

The matrix in which $m \neq n$.

- **Types of Rectangular Matrix**

(i) Row Matrix : When $m = 1, n \neq 1$.

e.g., $A = [1 \quad -1 \quad 2]_{1\times3}$

(ii) Column Matrix : When $m \neq 1, n = 1$.

e.g., $$A = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 3 \end{bmatrix}_{4\times1}$$

(iii) Vertical Matrix : When $m > n$.

e.g., $$A = \begin{bmatrix} 2 & 3 \\ 4 & 2 \\ -1 & 1 \end{bmatrix}_{3\times2}$$

(iv) Horizontal Matrix : When $n > m$.

e.g., $$A = \begin{bmatrix} 2 & -1 & 3 \\ 4 & 1 & 5 \end{bmatrix}_{2\times3}$$

1.5.2 Square Matrix

The matrix in which $m = n$.

Note : For square matrix, corresponding element of a_{ij} is a_{ji}.

- **Types of Square Matrix**

(i) Triangular Matrix :

(a) Upper Triangular Matrix : A square matrix is said to be upper triangular matrix if the elements below the diagonal are zero i.e. $a_{ij} = 0;\ i > j$

e.g., $$\begin{bmatrix} 6 & 2 & 3 \\ 0 & 5 & 8 \\ 0 & 0 & 4 \end{bmatrix}$$

(b) Lower Triangular Matrix :– A square matrix is said to be lower triangular matrix if all the elements above diagonal are zero i.e. $a_{ij} = 0;\ i < j$.

e.g., $$\begin{bmatrix} 6 & 0 & 0 \\ 2 & 5 & 0 \\ 3 & 6 & 4 \end{bmatrix}$$

Note : A triangular matrix is said to be strictly triangular matrix if all diagonal elements are zero.

Objective Questions

GATE PREVIOUS YEAR'S SOLVED QUESTIONS

1. The eigen vector(s) of the matrix

$$\begin{bmatrix} 0 & 0 & \alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \alpha \neq 0 \text{ is (are)}$$

(a) $(0, 0, \alpha)^T$ (b) $(\alpha, 0, 0)^T$

(c) $(0, 0, 1)^T$ (d) $(0, \alpha, 0)^T$

[GATE-1993]

2. If $A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & i & i \\ 0 & 0 & 0 & -i \end{bmatrix}$ then matrix A^4 calculated by the use of Cayley – Hamiton theorem (or) otherwise is

[GATE-1993]

3. If A and B are real symmetric matrices of order n then which of the following is true.

(a) $A A^T = I$ (b) $A = A^{-1}$

(c) $AB = BA$ (d) $(AB)^T = B^T A^T$

[GATE-1994 : CS]

4. The inverse of the matrix $\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ is

[GATE-1994]

5. A 5 × 7 matrix has all it entries equal to 1. Then the rank of a matrix is

(a) 7 (b) 5

(c) 1 (d) Zero

[GATE-1994 : EE]

6. The eigen values of the matrix $\begin{bmatrix} a & 1 \\ a & 1 \end{bmatrix}$ are

(a) $(a + 1), 0$ (b) $a, 0$

(c) $(a - 1), 0$ (d) $0, 0$

[GATE-1994 : EE]

7. The number of linearly independent solutions of the system of equations $\begin{bmatrix} 1 & 0 & 2 \\ 1 & -1 & 0 \\ 2 & -2 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$ is equal to

(a) 1 (b) 2

(c) 3 (d) 0

[GATE-1994 : EE]

8. The rank of (m × n) matrix (m < n) cannot be more than

(a) m (b) n

(c) mn (d) None

[GATE-1994 : EC]

9. Solve the following system

$$x_1 + x_2 + x_3 = 3$$
$$x_1 + x_3 = 0$$
$$x_1 - x_2 + x_3 = 1$$

(a) Unique solution

(b) No solution

(c) Infinite number of solutions

(d) Only one solution

[GATE-1994 : EC]

10. The rank of matrix $\begin{bmatrix} 0 & 0 & -3 \\ 9 & 3 & 5 \\ 3 & 1 & 1 \end{bmatrix}$ is

(a) 0 (b) 1

(c) 2 (d) 3

[GATE-1994 : CS]

11. The matrix $\begin{bmatrix} 1 & -4 \\ 1 & -5 \end{bmatrix}$ is an inverse of the matrix $\begin{bmatrix} 5 & -4 \\ 1 & -1 \end{bmatrix}$

(a) True (b) False

[GATE-1994 : PI]

12. If for a matrix, rank equals both the number of rows and number of columns, then the matrix is called

(a) Non-singular (b) singular
(c) transpose (d) minor

[GATE-1994 : PI]

13. The value of the following determinant $\begin{vmatrix} 1 & 4 & 9 \\ 4 & 9 & 16 \\ 9 & 16 & 25 \end{vmatrix}$ is

(a) 8 (b) 12
(c) –12 (d) –8

[GATE-1994 : PI]

14. For the following matrix $\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$ the number of positive roots is/are

(a) One (b) Two
(c) Four (d) No positive root

[GATE-1994 : PI]

15. Rank of the matrix $\begin{bmatrix} 0 & 2 & 2 \\ 7 & 4 & 8 \\ -7 & 0 & -4 \end{bmatrix}$ is 2

(a) True (b) False

[GATE-1994 : ME]

16. Find out the eigen values of the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 1 \\ 0 & 2 & 4 \end{bmatrix}$. For any one of the eigen values, find out the corresponding eigen vector?

[GATE-1994 : ME]

17. Given matrix $L = \begin{bmatrix} 2 & 1 \\ 3 & 2 \\ 4 & 5 \end{bmatrix}$ and $M = \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}$ then matrix product LM is

(a) $\begin{bmatrix} 8 & 1 \\ 13 & 2 \\ 22 & 5 \end{bmatrix}$ (b) $\begin{bmatrix} 6 & 5 \\ 9 & 8 \\ 12 & 13 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 8 \\ 2 & 13 \\ 5 & 22 \end{bmatrix}$ (d) $\begin{bmatrix} 6 & 2 \\ 9 & 4 \\ 0 & 5 \end{bmatrix}$

[GATE-1995 : PI]

18. Solve the system $2x + 3y + z = 9$, $4x + y = 7$, $x - 3y - 7z = 6$

[GATE-1995 : ME]

19. Among the following, the pair of the vector orthogonal to each other is

(a) [3, 4, 7], [3, 4, 7]
(b) [1, 0, 0], [1, 1, 0]
(c) [1, 0, 2], [0, 5, 0]
(d) [1, 1, 1], [-1, -1, -1]

[GATE-1995 : ME]

20. The inverse of the matrix $S = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ is

(a) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 1 & 1 \\ -1 & -1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 2 & -2 \\ -2 & 2 & -2 \\ 0 & 2 & 2 \end{bmatrix}$ (d) $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}$

[GATE-1995 : EE]

21. Given the matrix $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}$. Its eigen values are

[GATE-1995 : EE]

22. The rank of the following $(n + 1) \times (n + 1)$ matrix, where 'a' is a real number is

$$\begin{bmatrix} 1 & a & a^2 & \cdots & a^n \\ 1 & a & a^2 & \cdots & a^n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & a & a^2 & \cdots & a^n \end{bmatrix}$$

(a) 1
(b) 2
(c) n
(d) Depends on the value of a

[GATE-1995 : EE]

304. Consider the row vectors v = (1, 0) and w = (2, 0). The rank of the matrix $M = 2v^Tv + 3w^Tw$, where the superscript T denotes the transpose, is

(a) 1 (b) 2

(c) 3 (d) 4

[1 Mark : GATE-2021 (IN)]

305. The determinant of the matrix M shown below is

$$M = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 0 & 0 & 4 & 3 \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

[1 Mark : GATE-2021 (IN)]

306. Given $A = \begin{pmatrix} 2 & 5 \\ 0 & 3 \end{pmatrix}$. The value of the determinant $|A^4 - 5A^3 + 6A^2 + 2I| =$ ___________.

[2 Marks : GATE-2021 (IN)]

307. The rank of matrix $\begin{bmatrix} 1 & 2 & 2 & 3 \\ 3 & 4 & 2 & 5 \\ 5 & 6 & 2 & 7 \\ 7 & 8 & 2 & 9 \end{bmatrix}$ is

(a) 1 (b) 2

(c) 3 (d) 4

[1 Mark : GATE-2021 (CE) (Set-1)]

308. If $P = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $Q = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ then Q^TP^T is

(a) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$ (d) $\begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}$

[1 Mark : GATE-2021 (CE) (Set-1)]

309. The rank of the matrix $\begin{bmatrix} 5 & 0 & -5 & 0 \\ 0 & 2 & 0 & 1 \\ -5 & 0 & 5 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}$ is

(a) 1 (b) 2

(c) 3 (d) 4

[1 Mark : GATE-2021 (CE) (Set-2)]

310. If A is a square matrix then orthogonality property mandates

(a) $AA^T = I$ (b) $AA^T = 0$

(c) $AA^T = A^{-1}$ (d) $AA^T = A^2$

[1 Mark : GATE-2021 (CE) (Set-2)]

311. The smallest eigenvalue and the corresponding eigenvector of the matrix $\begin{bmatrix} 2 & -2 \\ -1 & 6 \end{bmatrix}$, respectively, are

(a) 1.55 and $\begin{Bmatrix} 2.00 \\ 0.45 \end{Bmatrix}$

(b) 2.00 and $\begin{Bmatrix} 1.00 \\ 1.00 \end{Bmatrix}$

(c) 1.55 and $\begin{Bmatrix} -2.55 \\ -0.45 \end{Bmatrix}$

(d) 1.55 and $\begin{Bmatrix} 2.00 \\ -0.45 \end{Bmatrix}$

[2 Marks : GATE-2021 (CE) (Set-2)]

312. Consider a vector p in 2-dimensional space. Let its direction (counter-clockwise angle with the positive x-axis) be θ. Let p be an eigenvector of a 2 × 2 matrix A with corresponding eigenvalue λ, λ > 0. If we denote the magnitude of a vector v by $\|v\|$, identify the VALID statement regarding p′, where p′ = Ap.

(a) Direction of p′ = λθ, $\|p'\| = \|p\|$

(b) Direction of p′ = θ, $\|p'\| = \lambda\|p\|$

(c) Direction of p′ = λθ, $\|p'\| = \lambda\|p\|$

(d) Direction of p′ = θ, $\|p'\| = \dfrac{\|p\|}{\lambda}$

[2 Marks : GATE-2021 (ME) (Set-1)]

313. Consider an n × n matrix A and a non-zero n × 1 vector p. Their product $Ap = \alpha^2 p$, where $\alpha \in \Re$ and $\alpha \notin \{-1, 0, 1\}$. Based on the given information,the eigen value of A^2 is :

(a) α (b) α^2

(c) $\sqrt{\alpha}$ (d) α^4

[1 Mark : GATE-2021 (ME) (Set-2)]

314. Consider the following matrix.

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

The largest eigenvalue of the above matrix is___________.

[2 Marks : GATE-2021 (CS) (Set-1)]

315. Suppose that P is a 4×5 matrix such that every solution of the equation $Px = 0$ is a scalar multiple of $[2\ 5\ 4\ 3\ 1]^T$. The rank of P is_________.

[1 Mark : GATE-2021 (CS) (Set-2)]

316. A, B, C and D are vectors of length 4.

$A = [a_1\ \ a_2\ \ a_3\ \ a_4]$

$B = [b_1\ \ b_2\ \ b_3\ \ b_4]$

$C = [c_1\ \ c_2\ \ c_3\ \ c_4]$

$D = [d_1\ \ d_2\ \ d_3\ \ d_4]$

It is known that B is not a scalar multiple of A. Also, C is linearly independent of A and B. Further, $D = 3A + 2B + C$.

The rank of the matrix $\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{bmatrix}$

is________.

[1 Mark : GATE-2021 (CH)]

317. Let A be a square matrix of size $n \times n$ $(n > 1)$. The elements of $A = \{a_{ij}\}$ are given by-

$$a_{ij} = \begin{cases} i \times j & \text{if } i \geq j \\ 0 & \text{if } i < j \end{cases}$$

The determinant of A is

(a) 0 (b) 1

(c) n! (d) $(n!)^2$

[2 Marks : GATE-2021 (CH)]

❍❍❍

ANSWERS AND EXPLANATIONS

1. ***Ans. (b & d)***

Given matrix is $A = \begin{bmatrix} 0 & 0 & \alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(i) Eigen values:–

Given matrix is an upper triangular matrix. Therefore diagonal elements of A are eigen values of A.

i.e. $\lambda = 0, 0, 0.$

(ii) Eigen vectors:–

Consider $(A - \lambda I)X = 0$

$$\Rightarrow \begin{bmatrix} 0-\lambda & 0 & \alpha \\ 0 & 0-\lambda & 0 \\ 0 & 0 & 0-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad ...(1)$$

Put $\lambda = 0$ in (1)

$$i.e. \begin{bmatrix} 0 & 0 & \alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad ...(2)$$

From eq. (2), we have

$$\alpha x_3 = 0$$

$$x_3 = 0$$

Therefore, any non zero vector with x_3 as zero is an eigen vector.

Hence, options (b) and (d) are correct.

2. ***Ans.* (I)**

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & i & i \\ 0 & 0 & 0 & -i \end{bmatrix}$$

$\Rightarrow$ Eigen values of A are $\lambda = 1, -1, i, -i$

The characteristic equation of A is

$$(\lambda - 1)(\lambda + 1)(\lambda - i)(\lambda + i) = 0$$

$$\Rightarrow (\lambda^2 - 1)(\lambda^2 + 1) = 0$$

$$\Rightarrow \lambda^4 - 1 = 0$$

By Cayley-Hamilton theorem we have $A^4 - I = 0$

$$\therefore A^4 = I$$

3. ***Ans. (d)***

By the reversal law of the transpose of the product of matrices, we have

$$(AB)^T = B^T A^T$$

4.
$$A^{-1} = \frac{adj(A)}{|A|}$$

$$|A| = \begin{vmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \end{vmatrix}$$

$$= 1(0 - 1) + 0 + 1(-1 - 0)$$

$$= -2$$

$$adj\ (A) = \begin{bmatrix} -1 & 1 & -1 \\ 0 & 0 & -2 \\ -1 & -1 & 1 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{adj(A)}{|A|}$$

$$= \frac{1}{(-2)} \begin{bmatrix} -1 & 1 & -1 \\ 0 & 0 & -2 \\ -1 & -1 & 1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 2 \\ 1 & 1 & -1 \end{bmatrix} \quad \textit{Ans.}$$

5. ***Ans. (c)***

Since all the elements of non-zero matrix are equal hence $\rho(A) = 1$.

6. ***Ans : (a)***

Given $A = \begin{bmatrix} a & 1 \\ a & 1 \end{bmatrix}$

$$\Rightarrow |A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} a-\lambda & 1 \\ a & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 - (a + 1)\lambda + 0 = 0$$

$\therefore$ $\lambda = 0, a + 1$ are the eigen values of A

7. ***Ans. (a)***

Given $\begin{bmatrix} 1 & 0 & 2 \\ 1 & -1 & 0 \\ 2 & -2 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & -1 & 0 \\ 2 & -2 & 0 \end{bmatrix}$$

$R_2 \to R_2 - R_1;\ R_3 \to R_3 - 2R_1$

$$\sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & -2 \\ 0 & -2 & -4 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$

$$\sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$r = \rho(A) = 2,$

$n = \text{number of variables} = 3$

$\therefore$ Number of linearly independent solutions

$= n - r = 3 - 2 = 1$

8. ***Ans. (a)***

We know that, $\rho(A_{mxn}) \le \min\{m, n\}$ but it is given that $m < n$

$\therefore \rho(A_{mxn}) \le m$

Hence $\rho(A_{m\times n})$ cannot be more than 'm'.

9. ***Ans. (b)***

Given $AX = B$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

Consider the augmented matrix [A|B]

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 0 & 1 & 0 \\ 1 & -1 & 1 & 1 \end{array}\right]$$

$R_2 \to R_2 - R_1$

$R_3 \to R_3 - R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -1 & 0 & -3 \\ 0 & -2 & 0 & -2 \end{array}\right]$$

$R_3 \to R_3 - 2R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & -1 & 0 & -3 \\ 0 & 0 & 0 & 4 \end{array}\right]$$

$\rho(A) = 2, \rho(A\,|\,B) = 3$

Here, $\rho(A) \ne \rho(A\,|\,B)$

$\therefore$ Solution does not exist.

10. ***Ans. (c)***

$$A = \begin{bmatrix} 0 & 0 & -3 \\ 9 & 3 & 5 \\ 3 & 1 & 1 \end{bmatrix}$$

$R_1 \leftrightarrow R_3$

$$\sim \begin{bmatrix} 3 & 1 & 1 \\ 9 & 3 & 5 \\ 0 & 0 & -3 \end{bmatrix}$$

$R_2 \to R_2 - 3R_1$

$$\sim \begin{bmatrix} 3 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & -3 \end{bmatrix}$$

$R_3 \to 2R_3 + 3R_2$

$$\sim \begin{bmatrix} 3 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore \quad \rho(A) = 2$

11. ***Ans. (a)***

Given $A = \begin{bmatrix} 5 & -4 \\ 1 & -1 \end{bmatrix}$

$\Rightarrow \quad A^{-1} = \dfrac{\text{adj}(A)}{|A|}$

$$= \frac{1}{-5+4}\begin{bmatrix} -1 & 4 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} 1 & -4 \\ 1 & -5 \end{bmatrix}$$

309. ***Ans. (c)***

$$A = \begin{bmatrix} 5 & 0 & -5 & 0 \\ 0 & 2 & 0 & 1 \\ -5 & 0 & 5 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 5 & 0 & -5 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix} (R_3 \to R_1 + R_3)$$

$$\sim \begin{bmatrix} 5 & 0 & -5 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.5 \end{bmatrix} \left(R_4 \to R_4 - \frac{R_2}{2}\right)$$

Which is the form of Echelon matrix.

So, the rank of matrix is the number of non-zero rows.

i.e., $\rho(A) = 3$

310. ***Ans. (a)***

A square matrix A is called an orthogonal matrix if the product of matrix A with its transpose matrix A^T is an identity matrix.

i.e., $AA^T = I$

311. ***Ans. (d)***

$$|A - \lambda I| = 0$$

$$\Rightarrow (2 - \lambda)(6 - \lambda) - 2 = 0$$

$$\Rightarrow \lambda^2 - 8\lambda + 10 = 0$$

$$\lambda = 4 \pm \sqrt{6}$$

Smallest eigen value

$$= 4 - \sqrt{6} = 1.55$$

$$\begin{bmatrix} 2-\lambda & -2 \\ -1 & 6-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2-\left(4-\sqrt{6}\right) & -2 \\ -1 & 6-\left(4-\sqrt{6}\right) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2+\sqrt{6} & -2 \\ -1 & 2+\sqrt{6} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-2x_1 + \sqrt{6}x_1 - 2x_2 = 0$$

$$\left(-2+\sqrt{6}\right)x_1 = 2x_2$$

$$x_1 = \frac{2x_2}{-2+\sqrt{6}}$$

$$\frac{x_1}{2} = \frac{x_2}{-2+\sqrt{6}} = \frac{x_2}{-0.45}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -0.45 \end{bmatrix}$$

312. ***Ans. (b)***

$\because$ A is a 2 × 2 matrix and P is the eigen vector of matrix A with corresponding eigen value λ.

Given, $P' = P$

$$AP = \lambda P$$

Hence, $P' = \lambda P$

$$\|P'\| = \|\lambda P\| = \lambda\|P\|$$

But direction of vector P′ will be same as vector P.

313. ***Ans. (d)***

Given, $AP = \alpha^2 P$

By comparision with

$$AX = \lambda X$$

$$\Rightarrow \lambda = \alpha^2$$

Hence, eigen value of A is α^2, so eigen value of A^2 is α^4.

314. ***Ans. 3***

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} -\lambda & 1 & 1 & 1 \\ 1 & -\lambda & 1 & 1 \\ 1 & 1 & -\lambda & 1 \\ 1 & 1 & 1 & -\lambda \end{vmatrix} = 0$$

$$C_1 \to C_1 + C_2 + C_3 + C_4$$

$$\Rightarrow \begin{vmatrix} -\lambda+3 & 1 & 1 & 1 \\ -\lambda+3 & -\lambda & 1 & 1 \\ -\lambda+3 & 1 & -\lambda & 1 \\ -\lambda+3 & 1 & 1 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow (-\lambda+3)\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -\lambda & 1 & 1 \\ 1 & 1 & -\lambda & 1 \\ 1 & 1 & 1 & -\lambda \end{vmatrix} = 0$$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 - R_1$
$R_4 \to R_4 - R_1$

$$\Rightarrow (-\lambda+3)\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & -\lambda-1 & 0 & 0 \\ 0 & 0 & -\lambda-1 & 0 \\ 0 & 0 & 0 & -\lambda-1 \end{vmatrix} = 0$$

or $(-\lambda + 3)(-\lambda - 1)^3 = 0$

$\Rightarrow \quad \lambda = 3, -1, -1, -1$

So, maximum eigen value is

$\lambda = 3$

315. ***Ans. 4***

$P_{4 \times 5} \Rightarrow$ Number of unknowns = 5

Also it is given that nullity of PX = 0 is 1

i.e., $\quad N(P) = 1$

Hence, Nullity = Number of unknowns – Rank

$1 = 5 - \rho(P)$

or $\quad \rho(P) = 5 - 1 = 4$

316. ***Ans. 3***

Since B is a not a scalar multiple of A

$\therefore$ B is linearly independent on A ...(1)

Given C is linearly independent on A ...(2)

and $\quad D = 3A + 2B + C$

From equation (1) and (2)

A, B, C are linearly independent and D is linearly dependent on A, B, C

$\therefore$ Number of linearly independent rows is 3

Hence, rank of matrix $\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{bmatrix}$ is 3.

317. ***Ans. (d)***

Square matrix of size $n \times n$ $(n > 1)$

Elements of A = $\{a_{ij}\}$

$$a_{ij} = \begin{cases} i \times j & \text{if } i \ge j \\ 0 & \text{if } i < j \end{cases}$$

Let, $\quad n = 2$

$$A_{2 \times 2} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 4 \end{bmatrix}$$

$\Rightarrow \quad |A| = (1 \times 2)^2 = (2!)^2$

$$A_{3 \times 3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 4 & 0 \\ 3 & 6 & 9 \end{bmatrix}$$

$|A| = 1 \times 4 \times 9$

$= (1 \times 2 \times 3)^2$

$= (3!)^2$

Similarly,

$|A_{n \times n}| = (n!)^2$

❍❍❍

Objective Questions PRACTICE SHEET

1. Evaluate $\begin{vmatrix} a & b+c & a^2 \\ b & c+a & b^2 \\ c & a+b & c^2 \end{vmatrix}$

(a) (a – b) (b – c) (c – a) (a + b + c)

(b) (a – b) (b – c) (a – c) (a + b + c)

(c) (a – b) (b – c) (c – a) (ab + bc + ca)

(d) (a – b) (b – c) (a – c) (ab + bc + ca)

2. If A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then choose the correct option below.

(a) $|\text{adj } A^{-1}| = |A|$ (b) A (adj A) = |A| I

(c) $|A^{-1}| = (|A|)^{-1}$ (d) Both (b) and (c)

3. If $\begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$, then for every positive integer n, A^n is equal to

(a) $\begin{bmatrix} 1+2n & 4n \\ n & 1+2n \end{bmatrix}$ (b) $\begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix}$

(c) $\begin{bmatrix} 1-2n & 4n \\ n & 1+2n \end{bmatrix}$ (d) None of these

4. Let H = $\begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix}$ and

$L = \begin{bmatrix} 1 & 0 & 0 \\ a & \frac{1}{2\sqrt{3}} & 0 \\ b & \frac{1}{2\sqrt{3}} & \frac{1}{6\sqrt{5}} \end{bmatrix}$. If $H = LL^T$, then

(a) a =1/3, b = 1/2

(b) a = 1/2, b = 1/3

(c) a = 1/3, b = 1/4

(d) a = 1/4, b = 1/3

5. The number of 3 × 3 non-singular matrices, with four entries as 1 and all other entries as 0, is

(a) 5 (b) 6

(c) at least 7 (d) < 4

6. If A = $\begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$, then matrix A is

(a) Idempotent

(b) Involutory

(c) Scalar

(d) Nilpotent

7. If A and B are square matrices of size n × n such that $A^2 - B^2 = (A - B)(A + B)$, then which of the following is always true?

(a) AB = BA

(b) either A or B is a zero matrix

(c) either A or B is an identity matrix

(d) A = B

8. There are two matrices A & B. Matrix A has x rows and x – 6 columns. Matrix B has y rows and y^2 columns. If commutativity holds for AB then the value of y is.

(a) 8 (b) 3

(c) 6 (d) 2

9. There are two matrices A & B. Matrix A has x rows and x – 6 columns. Matrix B has y rows and y^2 columns. If commutativity holds for AB then the value of x is.

(a) 8 (b) 3

(c) 6 (d) 9

10. If determinant of 3 × 3 matrix A is 2, then |3A| is equal to

(a) 9 (b) 27

(c) 54 (d) 81

ANSWERS AND EXPLANATIONS

1. *Ans. (b)*

$$\begin{vmatrix} a & b+c & a^2 \\ b & c+a & b^2 \\ c & a+b & c^2 \end{vmatrix}$$

$c_1 \to c_1 + c_2$

$$= \begin{vmatrix} a+b+c & b+c & a^2 \\ b+c+a & c+a & b^2 \\ c+a+b & a+b & c^2 \end{vmatrix}$$

$$= (a+b+c) \begin{vmatrix} 1 & b+c & a^2 \\ 1 & c+a & b^2 \\ 1 & a+b & c^2 \end{vmatrix}$$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 - R_1$

$$= \quad (a+b+c) \begin{vmatrix} 1 & b+c & a^2 \\ 0 & a-b & b^2-a^2 \\ 0 & a-c & c^2-a^2 \end{vmatrix};$$

$$= (a+b+c)\,(a-b)\,(a-c) \begin{vmatrix} 1 & b+c & a^2 \\ 0 & 1 & -(a+b) \\ 0 & 1 & -(c+a) \end{vmatrix}$$

$= (a + b + c)\ (a - b)\ (a - c)\ (a + b - a - c)$

$= (a - b)\ (b - c)\ (a - c)\ (a + b + c)$

i.e. option (b) is correct.

2. *Ans. (d)*

Given, $\quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}_{2\times2}$

We know, A(adj A)= |A| I

$\Rightarrow$ Option (b) is true.

Now, $\quad |A||\text{ adj } A| = ||A|\,I| = \begin{vmatrix} |A| & 0 \\ 0 & |A| \end{vmatrix}$

$= |A|^2$

$\Rightarrow \quad |\text{adj } A| = |A| \qquad (\because \text{A is } 2\times2)$

$\Rightarrow \quad |\text{Adj } A^{-1}| = |A^{-1}| \neq |A|$

$\Rightarrow$ option (a) is wrong

Again $\therefore \quad AA^{-1} = I$

$\Rightarrow \quad |AA^{-1}| = 1$

$\Rightarrow \quad |A|\ |A^{-1}| = 1$

if $\quad |A| \neq 0$

then $\quad |A^{-1}| = |A|^{-1}$

i.e. option (c) is true.

Hence most appropriate option is (d).

3. *Ans. (b)*

Given $\quad A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$

By observation, for n = 1 only

$$\begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} \text{ is true}$$

So, (a) & (c) cannot be correct options.

Let, $\quad A^n = \begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix}$

Now, $\quad A^{n+1} = A^n A$

$$= \begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix} \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 3+6n-4n & -4-8n+4n \\ 3n+1-zn & -4n-1+2n \end{bmatrix}$$

$$A^{n+1} = \begin{bmatrix} 1+2(n+1) & -4(n+1) \\ 3n+1-2n & 1-2(n+1) \end{bmatrix}$$

$\therefore$ By induction, $A^n = \begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix}$ is true

4. ***Ans. (b)***

Given $H = \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix}$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ a & \frac{1}{2\sqrt{3}} & 0 \\ b & \frac{1}{2\sqrt{3}} & \frac{1}{6\sqrt{5}} \end{bmatrix}$$

$$\text{Now, } LL^T = \begin{bmatrix} 1 & 0 & 0 \\ a & \frac{1}{2\sqrt{3}} & 0 \\ b & \frac{1}{2\sqrt{3}} & \frac{1}{6\sqrt{5}} \end{bmatrix} \begin{bmatrix} 1 & a & b \\ 0 & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} \\ 0 & 0 & \frac{1}{6\sqrt{5}} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & a & b \\ a & a^2 + \frac{1}{12} & ab + \frac{1}{12} \\ b & ab + \frac{1}{12} & b^2 + \frac{1}{12} + \frac{1}{180} \end{bmatrix}$$

Equality of matrices

$H = LL^T$

$\Rightarrow \quad a = \frac{1}{2}, \; b = \frac{1}{3}$ (comparing 1st rows)

5. ***Ans. (c)***

Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \text{ (4 entries as 1s)}$$

$\because \quad |A| = 1 \neq 0$

By observation, number of matrices with all principal diagonal entries as 1 are six. (*i.e.* no. of ways of placing remaining 1 at 6 non-diagonal positions.)

$\Rightarrow$ Either (b) or (c) is correct.

Also consider $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

$|B| = -1 \neq 0 \Rightarrow B$ is non-singular

i.e. at least 6 + 1 = 7 non-singular matrices.

Hence, option (C) is correct.

6. ***Ans. (d)***

Given $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$

$$A^2 = AA = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

i.e. A is nilpolent with index of nilpotency = 2.

7. ***Ans. (a)***

Given $A_{n\times n}$, $B_{n\times n}$ are square matrices.

s.t. $\quad A^2 - B^2 = (A - B)(A + B)$

$\Rightarrow \quad A^2 - B^2 = AA - BA + AB - BB$

$\Rightarrow \quad A^2 - B^2 = A^2 - B^2 + (AB - BA)$

Which is true only when AB – BA is a null matrix n × n.

i.e. $\quad AB - BA = 0$

$\Rightarrow \quad AB = BA$

8. ***Ans. (b)***

Principle of conformability of matrix multiplication: For matrices $A_{m\times n}$ & $B_{p\times q}$, the product AB is defined iff n = p & product BA is defined iff q = m.

Now, given $A_{x\times(x-6)}$ and $B_{y\times y^2}$ then for commutativity to hold, the necessary condition is (by principle of conformability of matrix multiplication)

$x - 6 = y \quad$ ...(i)

& $\quad y^2 = x \quad$ ...(ii)

$\Rightarrow \quad y^2 - 6 = y$

$\Rightarrow \quad y^2 - y - 6 = 0$

$(y - 3)(y + 2) = 0 \Rightarrow y = 3, -2$

i.e. $\quad y = 3, x = 9.$

9. ***Ans. (b)***

10. ***Ans. (c)***

Formula: If A is an n × n matrix & k is a scalar, then $\quad |kA| = k^n|A|$

Now, given $\quad |A| = 2, n = 3, k = 3$

$|3A| = 3^3 \times 2 = 54.$

11. ***Ans. (a)***

Given $a + b + c = 0$ ...(i)

Now, $\begin{vmatrix} a-\beta & c & b \\ c & b-\beta & a \\ b & a & c-\beta \end{vmatrix} = 0$ ...(ii)

$C_1 \to C_1 + C_2 + C_3$

$$\Rightarrow \begin{vmatrix} a+b+c-\beta & c & b \\ a+b+c-\beta & b-\beta & a \\ a+b+c-\beta & a & c-\beta \end{vmatrix} = 0$$

$$\Rightarrow -\beta \begin{vmatrix} 1 & c & b \\ 1 & b-\beta & a \\ 1 & a & c-\beta \end{vmatrix} = 0 \text{ using (i)}$$

$\Rightarrow \beta = 0$ is a solution of (ii)

12. ***Ans. (d)***

From the given matrix, we can write the characteristic equation as,

$$(\lambda - 1)(\lambda + 1)(\lambda - i)(\lambda + i) = 0$$

$$(\lambda^2 - 1)(\lambda^2 + 1) = 0$$

$$\lambda^4 - 1 = 0$$

Now according to Caley-Hamilton theorem,

$$A^4 - I = 0$$

or $A^4 = I$

13. ***Ans. (a)***

If, $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

then, $A^n = \begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix}$

Given $A = \begin{bmatrix} \cos\mu & \sin\mu \\ -\sin\mu & \cos\mu \end{bmatrix}$

$\therefore$ $A^2 = A$ (A is idempotent)

$$\Rightarrow \begin{bmatrix} \cos 2\mu & \sin 2\mu \\ -\sin 2\mu & \cos 2\mu \end{bmatrix} = \begin{bmatrix} \cos\mu & \sin\mu \\ -\sin\mu & \cos\mu \end{bmatrix} \quad \text{...(i)}$$

$\Rightarrow$ $\cos 2\mu = \cos\mu$

$\Rightarrow$ $2\mu = 2n\pi \pm \mu, n \in I$

$\Rightarrow$ $\mu = 2n\pi$ or $\dfrac{2n\pi}{3}$

For $n = 0, \mu = 0$ satisfies (i)

$n = 1, \mu = 2\pi$, satisfies (i)

$\mu = 2\pi/3$.

$$\left\{ \begin{aligned} \sin\frac{2\pi}{3} &= \frac{\sqrt{3}}{2} \\ \sin 2\left(\frac{2\pi}{3}\right) &= \frac{-\sqrt{3}}{2} \end{aligned} \right\}$$

$\Rightarrow$ $\sin 2\mu \neq \sin\mu$

$\mu = 2\pi/3$ does not satisfy (i)

Similarly, we can check other values of μ.

14. ***Ans. (b)***

Vectors $[1, 1, 1] = x_1$

$[1, a, a^2] = x_2$

$$[1\ 1\ 1] \begin{bmatrix} 1 \\ a \\ a^2 \end{bmatrix} = 1 + a + a^2 = 0$$

Hence, x_1, x_2 are orthogonal vectors.

15. ***Ans. (d)***

16. ***Ans. (c)***

Given $XY = O_{n\times n}$

$\Rightarrow$ $|XY| = 0$

$\Rightarrow$ $|X|\ |Y| = 0$

$\Rightarrow$ $|X| = 0$ or $|Y| = 0$ or both

17. ***Ans. (d)***

Since in an $M \times N$ matrix non-zero entries are in '*a*' rows & '*b*' columns such that no two are on the same row or column, this means that non-zero entries in '*a*' rows also, came in '*a*' columns, Which must be equal to '*b*', and viceversa. Hence maximum non zero entries are restricted by minimum of '*a*' & '*b*'.

18. ***Ans. (c)***

19. ***Ans. (c)***

$$A = [a_{ij}]_{m\times n},\ m < n.$$

under the given conditions

$$A = \begin{bmatrix} 1 & 2 & 3 & \cdots & n \\ 2\cdot 1 & 2\cdot 2 & 2\cdot 3 & \cdots & 2n \\ \vdots & \vdots & \vdots & & \vdots \\ (m-1)\cdot 1 & (m-1)\cdot 2 & (m-1)\cdot 3 & & (m-1)\cdot 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_i \to R_i - iR_1, 1 < i < m$

$$\sim \begin{bmatrix} 1 & 2 & 3 & \cdots & n \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

$\Rightarrow$ $\text{rank}(A) = 1$

20. ***Ans. (d)***

$A_{m\times x}$, $m > n$

Since two identical columns are there in A. This means there can be maximum (n – 1) independent columns in A.

Now, given that $m > n$

$\Rightarrow \quad (n-1) < m$

then we can get a minor of maximum order

$(n-1) \times (n-1)$

Hence, maximum possible rank of A is (n – 1).

21. ***Ans. (c)***

22. ***Ans. (a)***

The augmented matrix [A : B] of the given system of equation.

$$[A : B] = \left[\begin{array}{ccc|c} -2 & 1 & 1 & a \\ 1 & -2 & 1 & b \\ 1 & 1 & -2 & c \end{array}\right];$$

$R_1 \leftrightarrow R_3$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & -2 & c \\ 1 & -2 & 1 & b \\ -2 & 1 & 1 & a \end{array}\right];$$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 + 2R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & -2 & c \\ 0 & -3 & 3 & b-c \\ 0 & 3 & -3 & a+2c \end{array}\right];$$

$R_3 \to R_3 + R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & -2 & c \\ 0 & -3 & 3 & b-c \\ 0 & 0 & 0 & a+b+c \end{array}\right]$$

$\Rightarrow$ The third equation of the system is

$0x + 0y + 0z = (a + b + c)$

$\Rightarrow$ The system will not have any solution if

$(a + b + c) \neq 0$

23. ***Ans. (b)***

The given matrix has all rows identical.

Hence, rank = 1 (By definition of rank : all minors of order greater than 1 are zero & at least one minor of order 1 is non zero.)

24. ***Ans. (b)***

Given set of equations

$$x_1 + 2x_2 + x_3 + 4x_4 = 0 \quad \text{...(i)}$$

$$3x_1 + 3x_2 + 3x_3 + 12x_4 = 6 \quad \text{...(ii)}$$

From (ii), $\quad x_1 + 2x_2 + x_3 + 4x_4 = 2 \quad$ ...(iii)

Using (i), we get 0 = 2, which is not possible.

Note: rank method can also be applied.

rank(coeff. matrix) = 1 ≠ rank(aug. matrix) = 2

Hence, no solution.

25. ***Ans. (a)***

Q must have 4 L.I. rows & 4 L.I. columns.

26. ***Ans. (a)***

$$|P - \lambda I| = \begin{vmatrix} a-\lambda & 1 & 0 \\ 1 & a-\lambda & 1 \\ 0 & 1 & a-\lambda \end{vmatrix} = 0$$

$\Rightarrow \quad (a-\lambda)\{(a-\lambda)^2 - 1\} - \{a-\lambda\} = 0$

$\Rightarrow \quad (a-\lambda)\{(a-\lambda)^2 - 2\} = 0$

$\Rightarrow \quad \lambda = a, a \pm \sqrt{2}$

Note:

(i) In this question, all options (a), (b), (c), (d) satisfy sum of eigen values = trace of P = 3a.

(ii) But only (a) satisfyes det (P) = $a(a^2 - 2)$ = product of eigen values.

27. ***Ans. (b)***

(Elimination of options)

Trace(A) = 2 – 1 + 0 = 1

Sum of eigen values = 1, is satisfied by only option (b).

Note: Be careful it is only necessary condition but not sufficient.

28. ***Ans. (a)***

Consider the real symmetric matrix

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$$

Its eigen values are given by

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 \\ 2 & 3-\lambda \end{vmatrix} = 0$$

$\Rightarrow \quad 3 - 4\lambda + \lambda^2 - 4 = 0$

$\Rightarrow \quad \lambda = 2 \pm \sqrt{2}$

Conclusion: λ is real, one positive and other negative. Only option (a) satisfies the conclusion.

Note: General proof is not required.

29. ***Ans. (d)***

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 & 0 \\ 2 & 1-\lambda & 0 \\ 0 & 0 & -1-\lambda \end{vmatrix} = 0$$

$\Rightarrow \quad (-1-\lambda)\{(1-\lambda)^2-4\} = 0$

$\Rightarrow \quad (\lambda+1)\{(\lambda^2-2\lambda-3)\} = 0$

$\Rightarrow \quad \lambda = -1, -1, 3$

Observe that $|\lambda - 1| \le 2$ includes the above eigen values.

30. ***Ans. (d)***

Zero as an eigen value of a matrix A implies that

$|A - 0I| = 0$

$\Rightarrow \quad |A| = 0$

Now, given that

$PQ = I$

$\Rightarrow \quad |P|\,|Q| = 1$

$\Rightarrow$ Neither $|P| = 0$ nor $|Q| = 0$

$\Rightarrow$ Zero is neither eigen value of P nor Q.

31. ***Ans. (d)***

$$|A - \lambda I| = \begin{vmatrix} a^2-\lambda & ab & ac \\ ab & b^2-\lambda & bc \\ ac & bc & c^2-\lambda \end{vmatrix} = 0$$

$\Rightarrow (a^2 - \lambda)\{(b^2 - \lambda)(c^2 - \lambda) - b^2c^2\} - ab\{abc^2 - ab\lambda - abc^2\} + ac\,\{ab^2c - ab^2c + ac\lambda\} = 0$

$\Rightarrow (a^2-\lambda)\{-(b^2+c^2)\lambda + \lambda^2\}+a^2b^2\lambda+ a^2c^2\lambda = 0$

$\Rightarrow a^2\lambda^2 + (b^2+c^2)\,\lambda^2 - \lambda^3 = 0$

$\Rightarrow \lambda^2\,(a^2 + b^2 + c^2 - \lambda) = 0$

$\Rightarrow \lambda = 0, 0\ a^2 + b^2 + c^2$

Since a, b, c are non-zero real numbers.

Hence, $(a^2 + b^2 + c^2)$ is only non-zero eigen value.

32. ***Ans. (a)***

We know that

$AX = \lambda X\ (\because |A| \ne 0)$

$\Rightarrow \quad A^{-1}(AX) = A^{-1}(\lambda X)$

(Pre-multiplication)

$\Rightarrow \quad IX = \lambda A^{-1}X$

$A^{-1}X = \frac{1}{\lambda}X$

$\Rightarrow \frac{1}{\lambda}$ are eigen values of A^{-1}

$\therefore$ Eigen values of $A^{-1} = \frac{1}{2}, \frac{1}{4}, \frac{-1}{4}$

33. ***Ans. (b)***

We know that

$AX = \lambda X$

$kAX = k\lambda X$

$\Rightarrow \quad (kA)X = k\lambda X$

$\Rightarrow k\lambda$ are eigen value of kA.

$\therefore$ Eigen values of 2A are 2, –4 and 6.

34. ***Ans. (a)***

If λ is eigen value of A, then λ^n is eigen value of A^n

$\Rightarrow$ eigen values of $S^2 = 1^2, 5^2 = 1, 25$

35. ***Ans. (d)***

Let, $\quad P = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$

$\Rightarrow \quad P^{-1} = \frac{1}{3}\begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}$

$D = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$

We know,

$$A = \frac{1}{3}\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}$$

$$A = \frac{1}{3}\begin{bmatrix} 1 & 8 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 9 & 6 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$$